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Momentum-cutoff regularization and gauge invariance in QED

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Abstract

Based on a gauge-invariant form of the electron propagation function, we propose a formalism for QED which preserves its gauge-invariant character when both photon and electron propagators are regularized with a sharp momentum-cutoff procedure. Perturbation calculations of the regularized fermion effective action functional of an external electromagnetic field are given. We study radiative corrections induced by a momentum-cutoff vacuum and derive the corresponding Ward–Takahashi identity. Several problems encountered in an attempt of constructing a momentum-cutoff QED model are discussed.

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1. Introduction

Historically, the sharp momentum-cutoff method is the oldest procedure which has been used in QED in rendering the divergent loop integrals finite. In comparison with other more sophisticated regularization techniques (Pauli–Villars or dimensional methods), it has the advantage of relating the divergences to the short-distance behaviour of the theory and providing a basis for investigating the renormalization theory by using Wilson's approach [1]. Besides, this momentum-cutoff technique also exhibits practical significance in the studies of the dynamical chiral symmetry breaking [2] and even in recent issues of the dynamical electroweak symmetry breaking [3, 4]. However, the problem of gauge-invariance violation arising in computing divergent integrals with this method has made an impression for most people that the sharp momentum-cutoff procedure could not be used as an effective regulator in a gauge field theory. In this paper, we shall show that the difficulty of preserving gauge invariance of QED is not intrinsic to the sharp momentum-cutoff technique but can be overcome by implementing a minor formal modification of the conventional formalism of QED.

In addition to discussing a modified formalism of QED which admits a gauge-invariant momentum cutoff, this paper has a further purpose to illustrate the possibilities of constructing a momentum-cutoff QED model. According to the viewpoint prevailed in the modern quantum field theory, regulators are considered as purely formal means for handling divergent integrals in the renormalization procedure. Since all these divergent integrals will not appear in the renormalized theory, it is commonly believed that the resulting theory should not depend on the choice of the regulator if certain fundamental symmetries of the original theory have been properly respected in the regularization stage. However, the sharp momentum-cutoff regulator has a special feature that it enables us to make a realistic interpretation. In fact, from the mathematical point of view, a sharp momentum-cutoff procedure is nothing but imposing a boundary condition in the momentum space for all physical solutions. Thus, one could regard the momentum cutoff as a physical cutoff and study its effects on QED results [5, 6]. In this paper, we shall discuss a QED model which precludes all ultraviolet divergences by imposing boundary conditions at a neighbourhood of the infinite point of a 3D momentum space.

To show the way in which a sharp momentum-cutoff procedure could preserve the gauge invariance of QED theory, we consider the vacuum persistence amplitude in the presence of an external electromagnetic field $A_\mu(x)$ which, according to the conventional formalism of QED, can be expressed as a functional integral

$$\langle 0_+|0_-\rangle^A = \int [\mathrm{d}a] \exp\{\mathrm{i}(S[a] + W[A+a])\}, \tag{1.1}$$

where $a_\mu(x)$ denotes the fluctuations of the quantized photon fields with the action

$$S[a] = -\frac{1}{4}\int \mathrm{d}^4x [\partial_\mu a_\nu(x) - \partial_\nu a_\mu(x)][\partial^\mu a^\nu(x) - \partial^\nu a^\mu(x)] \tag{1.2}$$

and $W[A]$ represents the fermion effective action defined by

$$\mathrm{e}^{\mathrm{i}W[A]} = \int [\mathrm{d}\bar{\psi}\, \mathrm{d}\psi] \exp\left\{\mathrm{i}\int \mathrm{d}^4x \bar{\psi}(x)(\mathrm{i}\partial\!\!\!/ + eA\!\!\!/(x) - m_0)\psi(x)\right\}. \tag{1.3}$$

Since, in the momentum representation, a local gauge transformation of a_μ has the form

$$a_\mu(k) \to a_\mu(k) - \mathrm{i}k_\mu\lambda(k), \tag{1.4}$$

we could restrict the gauge transformations to a subgroup G_Λ defined by $\{\lambda(k) = 0$ for $|k| \geqslant \Lambda\}$ in the case that the functional integration on the right-hand side of equation (1.1) is carried out on the momentum-cutoff photon fields $a_\mu^\Lambda(a_\mu^\Lambda(k) \equiv a_\mu(k)$ for $|k| < \Lambda$, $a_\mu^\Lambda(k) \equiv 0$ for $|k| \geqslant \Lambda)$. In other words, a sharp momentum-cutoff regularization of photon fields will not destroy the gauge symmetry of the original theory. However, the gauge-invariance problem does arise in calculating $W[A]$ when one tries to regularize the fermion fields with a similar momentum-cutoff procedure.

In fact, by neglecting an irrelevant constant, from equation (1.3) we derive (see appendix D in [7])

$$W[A] = -\mathrm{eTr}\left\{\int_0^1 \mathrm{d}\lambda A\!\!\!/(x) S_\mathrm{F}[\lambda A]\right\}, \tag{1.5}$$

where

$$S_\mathrm{F}[A] = (\partial\!\!\!/ - \mathrm{i}eA\!\!\!/(x) + \mathrm{i}m_0)^{-1} \qquad \text{(with Feynman boundary conditions)} \tag{1.6}$$

is the Feynman electron propagator in an external field. Noting that, under the gauge transformation $A_\mu(x) \to A_\mu(x) + \partial_\mu\lambda(x)$, the Feynman propagator transforms as

$$S_\mathrm{F}[A] \to \mathrm{e}^{\mathrm{i}e\lambda(x)} S_\mathrm{F}[A]\, \mathrm{e}^{-\mathrm{i}e\lambda(x)}, \tag{1.7}$$

it is not difficult to give a formal proof of the gauge invariance of the effective action $W[A]$. Nevertheless, the expression of $W[A]$ given by equation (1.5) is mathematically ambiguous because the operator $\int_0^1 \mathrm{d}\lambda \not{A}(x) S_{\mathrm{F}}[\lambda A]$ is not a trace class operator on the Dirac field space $\mathcal{H}$. It follows that one must first regularize the propagator $S_{\mathrm{F}}[A]$ so that the right-hand side of equation (1.5) becomes a well-defined expression. A simplest way of regularizing $S_{\mathrm{F}}[A]$ is to restrict it in a momentum-cutoff Dirac field subspace $\mathcal{H}_\Lambda\{\psi^\Lambda|\psi^\Lambda(k) \equiv \psi(k)$ for $|k| < \Lambda, \psi^\Lambda(k) \equiv 0$ for $|k| \geqslant \Lambda\}$. Then the regularized effective action has the form

$$W_{S_\Lambda}[A] = -\mathrm{e}\mathrm{Tr}\left\{\int_0^1 \mathrm{d}\lambda \not{A}(x) S_\Lambda[\lambda A]\right\}, \tag{1.8}$$

where $S_\Lambda[A]$ denotes the restriction of $S_{\mathrm{F}}[A]$ on the momentum-cutoff subspace $\mathcal{H}_\Lambda$. Unfortunately, the momentum-cutoff regularized effective action $W_{S_\Lambda}[A]$ is no more a gauge-invariant functional since $\mathcal{H}_\Lambda$ is not a gauge-invariant subspace of Dirac fields for the gauge transformations in the subgroup G_Λ.

This difficulty can easily be overcome by replacing the gauge-covariant propagator $S_{\mathrm{F}}[A]$ with a gauge-invariant propagator $G[A]$. In fact, if we define the gauge-invariant propagator $G[A]$ by [8]

$$\langle x'|G[A]|x\rangle = \mathrm{e}^{-\mathrm{i}\mathrm{e}\chi(x',x)}\langle x'|S_{\mathrm{F}}[A]|x\rangle \tag{1.9}$$

with

$$\chi(x', x) = \int_0^1 A_\mu(x + s(x' - x))(x' - x)^\mu \,\mathrm{d}s, \tag{1.10}$$

we can rewrite equation (1.5) in the form

$$W[A] = -\mathrm{e}\int \mathrm{d}^4x \,\mathrm{tr}\left\{\int_0^1 \mathrm{d}\lambda \not{A}(x)\langle x|S_{\mathrm{F}}[\lambda A]|x\rangle\right\} = -\mathrm{e}\,\mathrm{Tr}\left\{\int_0^1 \mathrm{d}\lambda \not{A}(x) G[\lambda A]\right\}, \tag{1.11}$$

where the symbol tr denotes the trace taken on Dirac spinor indices. By regularizing the propagator $G[A]$ instead of $S[A]$, we obtain another momentum-cutoff regularized effective action

$$W_{G_\Lambda}[A] = -\mathrm{e}\,\mathrm{Tr}\left\{\int_0^1 \mathrm{d}\lambda \not{A}(x) G_\Lambda[\lambda A]\right\}. \tag{1.12}$$

Furthermore, we may define the transition amplitude for a momentum-cutoff vacuum as

$$\langle 0_+|0_-\rangle_\Lambda^A = \int [\mathrm{d}a^\Lambda] \exp\{\mathrm{i}(S[a] + W_{G_\Lambda}[A + a])\}, \tag{1.13}$$

where the symbol $\int \mathrm{d}[a^\Lambda]$ denotes a functional integration on the momentum-cutoff photon fields a_μ^Λ. It can be shown that both $W_{G_\Lambda}[A]$ and $\langle 0_+|0_-\rangle_\Lambda^A$ are well-defined gauge-invariant functionals of an external electromagnetic field $A_\mu(x)$ and they provide us a basis for the study of the momentum-cutoff model of QED.

In section 2, we study the perturbation expansions of two regularized versions of the fermion effective action $W_{S_\Lambda}[A]$ and $W_{G_\Lambda}[A]$ and calculate the vacuum polarization functions in second and fourth order. Section 3 presents a study of the radiative corrections induced by the momentum-cutoff quantum vacuum fluctuations. An alternative form of Ward–Takahashi identity which is valid for the momentum-cutoff vacuum has been derived. In section 4, we discuss two topics related to the momentum-cutoff model of QED, namely gauge-invariant formulation of quantized fermion field theory and Lorentz-invariance violation due to the ultraviolet cutoff.

2. Perturbation expansion of fermion effective action

We start by giving a perturbation expansion in the momentum representation for the Feynman electron propagator $S_{\mathrm{F}}[A]$

$$\langle p'|S_{\mathrm{F}}[A]|p\rangle = \int \mathrm{d}^4x \int \mathrm{d}^4x' \langle x'|S_{\mathrm{F}}[A]|x\rangle \exp^{\mathrm{i}(p'x'-px)}$$
$$= \sum_{n=0}^{\infty} \frac{(\mathrm{ie})^n}{(2\pi)^{4(n-1)}} \int \mathrm{d}^4k_1 \cdots \int \mathrm{d}^4k_n S^{(n)}_{\mu_1\cdots\mu_n}(p;k_1,\ldots,k_n)$$
$$\times A^{\mu_1}(k_1)\cdots A^{\mu_n}(k_n)\delta^{(4)}\left(p'-p-\sum_{j=1}^{n}k_j\right), \tag{2.1}$$

where

$$S^{(n)}_{\mu_1\cdots\mu_n}(p;k_1,\ldots,k_n) \equiv S_{\mathrm{F}}\left(p+\sum_{j=1}^{n}k_j\right)\gamma_{\mu_n}\cdots S_{\mathrm{F}}(p+k_1)\gamma_{\mu_1}S_{\mathrm{F}}(p) \tag{2.2}$$

with $S_{\mathrm{F}}(p) = \frac{\mathrm{i}(\not p+m_0)}{p^2-m_0^2+\mathrm{i}\epsilon}$. A similar perturbation expansion also exists for the gauge-invariant propagator $G[A]$

$$\langle p'|G[A]|p\rangle = \sum_{n=0}^{\infty} \frac{(\mathrm{ie})^n}{(2\pi)^{4(n-1)}} \int \mathrm{d}^4k_1 \cdots \int \mathrm{d}^4k_n G^{(n)}_{\mu_1\cdots\mu_n}(p;k_1,\ldots,k_n)$$
$$\times A^{\mu_1}(k_1)\cdots A^{\mu_n}(k_n)\delta^{(4)}\left(p'-p-\sum_{j=1}^{n}k_j\right). \tag{2.3}$$

To obtain an explicit expression of $G^{(n)}_{\mu_1\cdots\mu_n}(p;k_1,\ldots,k_n)$, we define the x-product $A \circ B$ of two operators A and B on the Dirac field space $\mathcal{H}$ by

$$\langle x',\alpha|A\circ B|x,\beta\rangle = \sum_{\gamma}\langle x',\alpha|A|x,\gamma\rangle\langle x',\gamma|B|x,\beta\rangle,$$

where α,β,γ denote the spinor indices. Since the function $\chi(x',x)$ given by equation (1.10) defines an operator on $\mathcal{H}$, $\langle x',\alpha|\chi|x,\beta\rangle = \delta_{\alpha\beta}\chi(x',x)$, the p-representation of the x-product of χ with an arbitrary operator B has the form

$$\langle p'|\chi\circ B|p\rangle = -\frac{\mathrm{i}}{(2\pi)^4}\int \mathrm{d}^4k \int_0^1 \mathrm{d}s A_\mu(k)\left(\frac{\partial}{\partial p'_\mu}+\frac{\partial}{\partial p_\mu}\right)\langle p'-k+sk|B|p+sk\rangle. \tag{2.4}$$

Thus, from equations (1.9), (2.1) and (2.3), we obtain

$$G^{(n)}_{\mu_1\cdots\mu_n}(p;k_1,\ldots,k_n)$$
$$= \sum_{m=0}^{n}\frac{\mathrm{i}^m}{m!}\int_0^1 \mathrm{d}s_1\cdots\int_0^1 \mathrm{d}s_m \partial_{\mu_1}\cdots\partial_{\mu_m} S^{(n-m)}_{\mu_{m+1}\cdots\mu_n}\left(p+\sum_{j=1}^{m}s_jk_j;k_{m+1},\ldots,k_n\right), \tag{2.5}$$

where ∂_μ denotes $\partial/\partial p^\mu$.

From equations (1.8) and (2.1), we obtain the perturbation expansion of the fermion effective action $W_{S_\Lambda}[A]$ in the momentum representation, which has the form

$$W_{S_\Lambda}[A] = -\frac{e}{(2\pi)^8}\int_{\mathcal{D}(\Lambda)} \mathrm{d}^4 p \int_{\mathcal{D}(\Lambda)} \mathrm{d}^4 p' \int_0^1 \mathrm{d}\lambda\, \mathrm{tr}\{\not{A}(p-p')\langle p'|S_{\mathrm{F}}[\lambda A]|p\rangle\}$$
$$= \sum_{n=1}^{\infty} W_{S_\Lambda}^{(n)}[A], \tag{2.6}$$

$$W_{S_\Lambda}^{(n)}[A] = \frac{\mathrm{i}(\mathrm{i}e)^n}{n(2\pi)^{4n}}\int \mathrm{d}^4 k_1 \cdots \int \mathrm{d}^4 k_n A^{\mu_1}(k_1)\cdots A^{\mu_n}(k_n)\delta^{(4)}\left(\sum_{j=1}^n k_j\right)$$
$$\times \int_{\mathcal{D}(\Lambda,k_n)} \mathrm{d}^4 p\, \mathrm{tr}\big[S^{(n-1)}_{\mu_1\cdots\mu_{n-1}}(p;k_1,\ldots,k_{n-1})\gamma_{\mu_n}\big], \tag{2.7}$$

where $\mathcal{D}(\Lambda)$ denotes a momentum-integration domain and $\mathcal{D}(\Lambda,k) = \{p \mid p \in \mathcal{D}(\Lambda)$ and $p-k \in \mathcal{D}(\Lambda)\}$. Similarly, for the gauge-invariant effective action $W_{G_\Lambda}[A]$, we have

$$W_{G_\Lambda}[A] = \sum_{n=1}^{\infty} W_{G_\Lambda}^{(n)}[A],$$

$$W_{G_\Lambda}^{(n)}[A] = \frac{\mathrm{i}(\mathrm{i}e)^n}{n(2\pi)^{4n}}\int \mathrm{d}^4 k_1 \cdots \int \mathrm{d}^4 k_n A^{\mu_1}(k_1)\cdots A^{\mu_n}(k_n)\delta^{(4)}\left(\sum_{j=1}^n k_j\right)$$
$$\times \int_{\mathcal{D}(\Lambda,k_n)} \mathrm{d}^4 p\, \mathrm{tr}\big[G^{(n-1)}_{\mu_1\cdots\mu_{n-1}}(p;k_1,\ldots,k_{n-1})\gamma_{\mu_n}\big]. \tag{2.8}$$

Now we should discuss the momentum-integration domains $\mathcal{D}(\Lambda)$ in more detail. We first take the point of view that the restriction of the integration domain is due to a cutoff imposed on the Dirac spinor fields. Let $\mathcal{D}(\Lambda)$, $\Lambda \in (0,\infty)$, be a closed region in the Minkowski 4-momentum space $\mathcal{R}^4\{p_\mu;\, \mu = 0,1,2,3\}$ which satisfies the following conditions:

$$|p_0|^2 + |\mathbf{p}|^2 < \Lambda^2 \Longrightarrow p \in \mathcal{D}(\Lambda), \tag{2.9a}$$
$$\int_{\mathcal{D}(\Lambda)} \mathrm{d}^4 p \big|\big(p^2 - m_0^2 + \mathrm{i}\varepsilon\big)^{-1}\big| < +\infty, \tag{2.9b}$$
$$p \in \mathcal{D}(\Lambda) \Longrightarrow -p \in \mathcal{D}(\Lambda). \tag{2.9c}$$

Condition (2.9b) is imposed to ensure that all the p-integrals appeared on the right-hand sides of equations (2.6) and (2.8) against ultraviolet divergence. Since Lorentz invariance is not compatible with condition (2.9b), the most natural cutoff is a 3D momentum cutoff with rotational symmetry in a given inertial system (see [5, 6]). In the following, we shall denote this particular cutoff by

$$\mathcal{D}_3(\Lambda) = \mathcal{R}^4\{p_\mu \mid |\mathbf{p}| < \Lambda\}. \tag{2.10}$$

In order to preserve Lorentz symmetry, one could consider another cutoff which is imposed on the Euclidian 4-momenta obtained after a Wick rotation of the p_0-axis, i.e. $\mathcal{D}(\Lambda)$ are closed regions in the Euclidean 4-momentum space $\mathcal{R}^4\{p_\alpha;\, \alpha = 1,2,3,4\}$ with $p_4 = -\mathrm{i}p_0$. Then we have a Lorentz-invariant cutoff of Euclidian 4-momenta which we shall denote as

$$\mathcal{D}_4(\Lambda) = \mathcal{R}^4\{p_\alpha \mid |p| < \Lambda\}, \tag{2.11}$$

where $|p|^2 = |\mathbf{p}|^2 + p_4^2 = -p^2$ is a Lorentz invariant. Since any sharp cutoff on the p_0-axis destroys the analyticity of all functions about p_0, one would obtain wrong results in calculating integrals on a p_0-cutoff domain by using the Wick rotation technique. Thus, we should not regard the Lorentz-invariant cutoff $\mathcal{D}_4(\Lambda)$ as a cutoff imposed on the Dirac spinor fields.

However, as a formal means of rendering the divergent integrals finite, this momentum-cutoff regulator has been used in many cases for the studies of quantum field theory. In the following, we shall use the cutoff $\mathcal{D}(\Lambda)$ in both senses (i.e. before or after the Wick rotation) and always require it to satisfy conditions (2.9).

After having given a precise definition of the cutoff $\mathcal{D}(\Lambda)$, we now turn to deriving some general properties of the momentum-cutoff regularized effective actions $W_{S_\Lambda}^{(n)}[A]$ and $W_{G_\Lambda}^{(n)}[A]$. Introduce the symmetrized functions

$$\bar{S}^{(n)}_{\mu_1\cdots\mu_n}(p;k_1,\ldots,k_n) = (n!)^{-1}\sum_{\pi\{1,\ldots,n\}} S^{(n)}_{\mu_1\cdots\mu_n}(p;k_1,\ldots,k_n), \tag{2.12}$$

$$\bar{G}^{(n)}_{\mu_1\cdots\mu_n}(p;k_1,\ldots,k_n) = (n!)^{-1}\sum_{\pi\{1,\ldots,n\}} G^{(n)}_{\mu_1\cdots\mu_n}(p;k_1,\ldots,k_n), \tag{2.13}$$

where $\pi\{1,\ldots,n\}$ denotes all the permutations on the n-point set $\{(\mu_j,k_j) \mid j=1,\ldots,n\}$. For these symmetrized functions, we find the following identities (see the appendix for a proof):

$$\begin{aligned}&\mathrm{tr}\big[\bar{S}^{(n-1)}_{\mu_1\cdots\mu_{n-1}}(p;k_1,\ldots,k_{n-1})\gamma_{\mu_n}\big]\\ &\qquad = (-1)^n\,\mathrm{tr}\left[\bar{S}^{(n-1)}_{\mu_1\cdots\mu_{n-1}}\left(-p-\sum_{j=1}^{n-1}k_j;k_1,\ldots,k_{n-1}\right)\gamma_{\mu_n}\right],\end{aligned} \tag{2.14}$$

$$\begin{aligned}&\mathrm{tr}\big[\bar{G}^{(n-1)}_{\mu_1\cdots\mu_{n-1}}(p;k_1,\ldots,k_{n-1})\gamma_{\mu_n}\big]\\ &\qquad = (-1)^n\,\mathrm{tr}\left[\bar{G}^{(n-1)}_{\mu_1\cdots\mu_{n-1}}\left(-p-\sum_{j=1}^{n-1}k_j;k_1,\ldots,k_{n-1}\right)\gamma_{\mu_n}\right],\end{aligned} \tag{2.15}$$

$$\begin{aligned}\mathrm{i}nk^\mu\bar{S}^{(n)}_{\mu\mu_1\cdots\mu_{n-1}}(p;k,k_1,\ldots,k_{n-1}) &= \bar{S}^{(n-1)}_{\mu_1\cdots\mu_{n-1}}(p+k;k_1,\ldots,k_{n-1})\\ &\quad - \bar{S}^{(n-1)}_{\mu_1\cdots\mu_{n-1}}(p;k_1,\ldots,k_{n-1}),\end{aligned} \tag{2.16}$$

$$k_1^{\mu_1}\bar{G}^{(n)}_{\mu_1\cdots\mu_n}(p;k_1,\ldots,k_n) = 0. \tag{2.17}$$

Then we obtain the following.

Proposition 1. $W_{S_\Lambda}^{(2n+1)}[A] = W_{G_\Lambda}^{(2n+1)}[A] = 0, \forall n = 0,1,\ldots$ *(Furry's theorem).*

Proposition 2. $W_{G_\Lambda}^{(n)}[A], n = 1,2,\ldots,$ *are gauge-invariant functionals of* A_μ.

Proposition 3. *For* $n > 4$, $\lim_{\Lambda\to\infty} W_{S_\Lambda}^{(n)}[A] = \lim_{\Lambda\to\infty} W_{G_\Lambda}^{(n)}[A] = W^{(n)}[A]$ *exist. They are both gauge invariant and Lorentz invariant.*

Proposition 1 can easily be derived from the identities (2.14), (2.15) if we note that from (2.9c) we have $p\in\mathcal{D}(\Lambda,k_n) \Longrightarrow -p+k_n\in\mathcal{D}(\Lambda,k_n)$. Proposition 2 is a direct consequence of (2.17). For a proof of proposition 3, we note that the p-integrals for $S^{(n)}_{\mu_1\cdots\mu_n}(p;k_1,\ldots,k_n)$ and $G^{(n)}_{\mu_1\cdots\mu_n}(p;k_1,\ldots,k_n)$ converge if $n>3$ and therefore independent of the regularization procedure. Besides, from equations (2.1) and (2.5), we see that the difference between two types of functions $S^{(n)}(p)$ and $G^{(n)}(p)$ has the form $\partial_\mu F(p)$, which yield vanishing contributions after p-integration when $n>3$.

In the following, we shall calculate $W_{S_\Lambda}^{(n)}[A]$ and $W_{G_\Lambda}^{(n)}[A]$ for $n\leqslant 4$ and study their asymptotic behaviour when $\Lambda\to\infty$, in particular the effects caused by different modes of the momentum-cutoff procedure. To this end, we rewrite the effective action $W^{(n)}_{G_\Lambda}[A]$ in the form

$$W^{(n)}_{G_\Lambda}[A] = \frac{1}{n(2\pi)^4}\int \mathrm{d}^4k_1\cdots\int \mathrm{d}^4k_n A^{\mu_1}(k_1)\cdots A^{\mu_n}(k_n)\delta^{(4)}\left(\sum_{j=1}^{n}k_j\right)K^{\Lambda}_{\mu_1\cdots\mu_n}(k_1,\ldots,k_n), \tag{2.18}$$

where we have introduced the symmetrized gauge-invariant vacuum polarization functions

$$K^{\Lambda}_{\mu_1\cdots\mu_n}(k_1,\ldots,k_n) = \frac{\mathrm{i}(\mathrm{ie})^n}{(2\pi)^{4(n-1)}n!}\sum_{\pi\{1,\ldots,n\}} \times\int_{\mathcal{D}(\Lambda,k_n)}\mathrm{d}^4p\,\mathrm{tr}\big[G^{(n-1)}_{\mu_1\cdots\mu_{n-1}}(p;k_1,\ldots,k_{n-1})\gamma_{\mu_n}\big], \tag{2.19}$$

so that the vacuum polarization current density can be expressed as

$$J_\mu(k,A) = -(2\pi)^4\frac{\delta W_{G_\Lambda}[A]}{\delta A^\mu(-k)} = -\sum_{n=0}^{\infty}\int \mathrm{d}^4k_1\cdots\int \mathrm{d}^4k_n K^{\Lambda}_{\mu\mu_1\cdots\mu_n}(-k,k_1,\ldots,k_n) \times A^{\mu_1}(k_1)\cdots A^{\mu_n}(k_n)\delta^{(4)}\left(k-\sum_{j=1}^{n}k_j\right). \tag{2.20}$$

Furthermore, we write $K^{\Lambda}_{\mu_1\cdots\mu_n}(k_1,\ldots,k_n)$ as a sum of an ordinary term and a gauge-correction term

$$K^{\Lambda}_{\mu_1\cdots\mu_n}(k_1,\ldots,k_n) = \Pi^{\Lambda}_{\mu_1\cdots\mu_n}(k_1,\ldots,k_n) + \Delta^{\Lambda}_{\mu_1\cdots\mu_n}(k_1,\ldots,k_n), \tag{2.21}$$

where

$$\Pi^{\Lambda}_{\mu_1\cdots\mu_n}(k_1,\ldots,k_n) = \frac{\mathrm{i}(\mathrm{ie})^n}{(2\pi)^{4(n-1)}n!}\sum_{\pi\{1,\ldots,n\}} \times\int_{\mathcal{D}(\Lambda,k_n)}\mathrm{d}^4p\,\mathrm{tr}\big[S^{(n-1)}_{\mu_1\cdots\mu_{n-1}}(p;k_1,\ldots,k_{n-1})\gamma_{\mu_n}\big], \tag{2.22}$$

$$\Delta^{\Lambda}_{\mu_1\cdots\mu_n}(k_1,\ldots,k_n) = \frac{\mathrm{i}(\mathrm{ie})^n}{(2\pi)^{4(n-1)}n!}\sum_{\pi\{1,\ldots,n\}}\sum_{m=1}^{n-1}\frac{\mathrm{i}^m}{m!}\int_0^1\mathrm{d}s_1\cdots\int_0^1\mathrm{d}s_m \times\int_{\mathcal{D}(\Lambda,k_n)}\mathrm{d}^4p\,\partial_{\mu_1}\cdots\partial_{\mu_m}\,\mathrm{tr}\left[S^{(n-m-1)}_{\mu_{m+1}\cdots\mu_{n-1}}\left(p+\sum_{j=1}^{m}s_jk_j;k_{m+1},\ldots,k_{n-1}\right)\gamma_{\mu_n}\right]. \tag{2.23}$$

2.1. *Calculation of $W^{(2)}_{S_\Lambda}[A]$ and $W^{(2)}_{G_\Lambda}[A]$*

From equations (2.22) and (2.23), we obtain

$$\Pi^{\Lambda}_{\mu\nu}(-k,k) = \frac{-\mathrm{ie}^2}{2(2\pi)^4}\left\{\int_{\mathcal{D}(\Lambda,k)}\mathrm{d}^4p\,\mathrm{tr}[S_{\mathrm{F}}(p-k)\gamma_\mu S_{\mathrm{F}}(p)\gamma_\nu] + \int_{\mathcal{D}(\Lambda,-k)}\mathrm{d}^4p\,\mathrm{tr}[S_{\mathrm{F}}(p+k)\gamma_\mu S_{\mathrm{F}}(p)\gamma_\nu]\right\}, \tag{2.24}$$

$$\Delta^{\Lambda}_{\mu\nu}(-k,k) = \frac{\mathrm{i}\,e^2}{2(2\pi)^4}\int_0^1 \mathrm{d}s \left\{ \int_{\mathcal{D}(\Lambda,k)} \mathrm{d}^4p\,\mathrm{tr}[S_{\mathrm{F}}(p-sk)\gamma_\mu S_{\mathrm{F}}(p-sk)\gamma_\nu] + \int_{\mathcal{D}(\Lambda,-k)} \mathrm{d}^4p\,\mathrm{tr}[S_{\mathrm{F}}(p+sk)\gamma_\mu S_{\mathrm{F}}(p+sk)\gamma_\nu] \right\}. \tag{2.25}$$

To study the asymptotic behaviour of the polarization function $K^{\Lambda}_{\mu\nu}(-k,k)$, we write $S_{\mathrm{F}}(p+q)$ appearing on the right-hand sides of equations (2.24) and (2.25) in a power series of q,

$$S_{\mathrm{F}}(p+q) = S_{\mathrm{F}}(p) + S^{(1)}_{\mathrm{F}}(p,q) + S^{(2)}_{\mathrm{F}}(p,q) + O(q^3), \tag{2.26}$$

where

$$S^{(1)}_{\mathrm{F}}(p,q) = \frac{\mathrm{i}\not{q} - 2(qp)S_{\mathrm{F}}(p)}{p^2-m_0^2}, \tag{2.27}$$

$$S^{(2)}_{\mathrm{F}}(p,q) = \frac{4(qp)^2S_{\mathrm{F}}(p) - 2\mathrm{i}(qp)\not{q}}{\left(p^2-m_0^2\right)^2} - \frac{q^2S_{\mathrm{F}}(p)}{p^2-m_0^2}. \tag{2.28}$$

By implementing the expansion (2.26), it can clearly be seen that the zeroth and first degree terms in $\Pi^{\Lambda}_{\mu\nu}(-k,k)$ and $\Delta^{\Lambda}_{\mu}(-k,k)$ cancel out completely so that the remaining terms in $K^{\Lambda}_{\mu\nu}(-k,k)$ are at most logarithmically divergent. Furthermore, we can now safely replace the integration domain $\mathcal{D}(\Lambda,k)$ in the expression of $K^{\Lambda}_{\mu\nu}(-k,k)$ with $\mathcal{D}(\Lambda)$, because for logarithmically divergent integrals the error produced by this replacement is of the order $O(\Lambda^{-1})$. Thus, the polarization function can be written as

$$K^{\Lambda}_{\mu\nu}(-k,k) = \Pi^{(2)}_{\mu\nu}(\Lambda,k) + \Delta^{(2)}_{\mu\nu}(\Lambda,k) + \Pi^{R}_{\mu\nu}(k) + \Delta^{R}_{\mu\nu}(k) + O(\Lambda^{-1}), \tag{2.29}$$

where

$$\Pi^{(2)}_{\mu\nu}(\Lambda,k) = \frac{-\mathrm{i}e^2}{(2\pi)^4}\int_{\mathcal{D}(\Lambda)} \mathrm{d}^4p\,\mathrm{tr}\big[S^{(2)}_{\mathrm{F}}(p,k)\gamma_\mu S_{\mathrm{F}}(p)\gamma_\nu\big], \tag{2.30}$$

$$\Delta^{(2)}_{\mu\nu}(\Lambda,k) = \frac{\mathrm{i}e^2}{3(2\pi)^4}\int_{\mathcal{D}(\Lambda)} \mathrm{d}^4p\,\mathrm{tr}\big[S^{(1)}_{\mathrm{F}}(p,k)\gamma_\mu S^{(1)}_{\mathrm{F}}(p,k)\gamma_\nu + 2S^{(2)}_{\mathrm{F}}(p,k)\gamma_\mu S_{\mathrm{F}}(p)\gamma_\nu\big] \tag{2.31}$$

and $\Pi^{R}_{\mu\nu}(k)$ and $\Delta^{R}_{\mu\nu}(k)$ denote the second degree remainders in the Taylor expansion of $\Pi^{\Lambda}_{\mu\nu}(-k,k)$ and $\Delta^{\Lambda}_{\mu}(-k,k)$, respectively. Since the expansion coefficients of these remainder functions are all expressed with convergent integrals, $\Pi^{R}_{\mu\nu}(k)$ and $\Delta^{R}_{\mu\nu}(k)$ are independent of the regulator used in calculation. In fact, our result agrees with the well-known form of the renormalized polarization function derived from the conventional QED formalism, i.e.

$$\Delta^{R}_{\mu\nu}(k) = 0, \qquad \Pi^{R}_{\mu\nu}(k) = (k^2g_{\mu\nu} - k_\mu k_\nu)\Pi^c(k^2) \qquad \text{with} \qquad \Pi^c(0) = 0. \tag{2.32}$$

On the other hand, the second degree terms $\Pi^{(2)}_{\mu\nu}(\Lambda,k)$ and $\Delta^{(2)}_{\mu\nu}(\Lambda,k)$ do depend on the regulator. Let

$$\Pi^{\mathrm{div}}_{\mu\nu}(\Lambda,k) = \Pi^{(2)}_{\mu\nu}(\Lambda,k) + \Delta^{(2)}_{\mu\nu}(\Lambda,k). \tag{2.33}$$

Then the polarization function has the form

$$K^{\Lambda}_{\mu\nu}(-k,k) = \Pi^{\mathrm{div}}_{\mu\nu}(\Lambda,k) + (k^2g_{\mu\nu} - k_\mu k_\nu)\Pi^c(k^2) + O(\Lambda^{-1}). \tag{2.34}$$

Simple trace calculations show that

$$\Pi^{\rm div}_{\mu\upsilon}(\Lambda,k)=\frac{8{\rm ie}^2}{3(2\pi)^4}\int_{\mathcal{D}(\Lambda)}{\rm d}^4p\left\{\frac{k^2g_{\mu\nu}-k_\mu k_\nu}{\left(p^2-m_0^2\right)^2}-\frac{(pk)^2g_{\mu\nu}-(pk)(p_\mu k_\nu+k_\mu p_\nu)+k^2p_\mu p_\nu}{\left(p^2-m_0^2\right)^3}\right\}. \tag{2.35}$$

If we use the Lorentz-invariant cutoff, i.e. let $\mathcal{D}(\Lambda)=\mathcal{D}_4(\Lambda)$, we obtain

$$\Pi^{\rm div}_{\mu\upsilon}(\Lambda,k)=\Pi(\Lambda)(k^2g_{\mu\nu}-k_\mu k_\nu), \tag{2.36}$$

with

$$\Pi(\Lambda)=-\frac{e^2}{12\pi^2}\left[\ln\frac{\Lambda^2}{m_0^2}-\frac{1}{2}\right]+O(\Lambda^{-1}). \tag{2.37}$$

However, if we use the cutoff $\mathcal{D}_3(\Lambda)$ as our regulator, the divergent term $\Pi^{\rm div}_{\mu\upsilon}(\Lambda,k)$ will no more be a Lorentz tensor. Let us specify the inertial system in which $\mathcal{D}_3(\Lambda)$ is implemented with a timelike unit vector n^μ, then we have

$$\Pi^{\rm div}_{\mu\upsilon}(\Lambda,k)=\Pi'(\Lambda)(k^2g_{\mu\nu}-k_\mu k_\nu)+\delta\Pi^{\rm div}_{\mu\upsilon}(\Lambda,k), \tag{2.38}$$

where

$$\Pi'(\Lambda)=-\frac{e^2}{12\pi^2}\left[\ln\frac{\Lambda^2}{m_0^2}+2\ln 2-\frac{4}{3}\right]+O(\Lambda^{-1}), \tag{2.39}$$

$$\delta\Pi^{\rm div}_{\mu\upsilon}(\Lambda,k)=a[k^2n_\mu n_\nu+(kn)^2g_{\mu\nu}-(kn)(k_\mu n_\nu+k_\nu n_\mu)],$$
$$\text{with}\quad a=\frac{e^2}{36\pi^2}+O(\Lambda^{-1}). \tag{2.40}$$

We shall use the term 'Lorentz anomaly' to denote the violation of Lorentz invariance caused by the ultraviolet cutoff in a momentum-cutoff model of quantum field theory and discuss the related problems in section 4.

2.2. *Calculation of* $W^{(4)}_{S_\Lambda}[A]$ *and* $W^{(4)}_{G_\Lambda}[A]$

Since $\Pi^\Lambda_{\mu_1\mu_2\mu_3\mu_4}(k_1,k_2,k_3,k_4)$ and $\Delta^\Lambda_{\mu_1\mu_2\mu_3\mu_4}(k_1,k_2,k_3,k_4)$ contain at most logarithmically divergences, we could replace the momentum-integration domain $\mathcal{D}(\Lambda,k)$ with $\mathcal{D}(\Lambda)$, i.e.

$$\Pi^\Lambda_{\mu_1\mu_2\mu_3\mu_4}(k_1,k_2,k_3,k_4)=\frac{{\rm ie}^4}{(2\pi)^{12}4!}\sum_{\pi\{1,\ldots,4\}}\times\int_{\mathcal{D}(\Lambda)}{\rm d}^4p\,{\rm tr}\big[S^{(3)}_{\mu_1\mu_2\mu_3}(p;k_1,k_2,k_3)\gamma_{\mu_4}\big]+O(\Lambda^{-1}), \tag{2.41}$$

$$\Delta^\Lambda_{\mu_1\mu_2\mu_3\mu_4}(k_1,k_2,k_3,k_4)=\frac{{\rm ie}^4}{(2\pi)^{12}4!}\sum_{\pi\{1,\ldots,4\}}\int_{\mathcal{D}(\Lambda)}{\rm d}^4p\sum_{m=1}^{3}\frac{{\rm i}^m}{m!}\int_0^1{\rm d}s_1\cdots\int_0^1{\rm d}s_m\times\partial_{\mu_1}\cdots\partial_{\mu_m}{\rm tr}\left[S^{(3-m)}_{\mu_{m+1}\cdots\mu_3}\left(p+\sum_{j=1}^{m}s_jk_j;k_{m+1},\ldots,k_3\right)\gamma_{\mu_4}\right]+O(\Lambda^{-1}). \tag{2.42}$$

From

$$\frac{\partial}{\partial(k_j)_\mu}\Delta^\Lambda_{\mu_1\mu_2\mu_3\mu_4}(k_1,k_2,k_3,k_4)=O(\Lambda^{-1}),\qquad j=1,2,3,4, \tag{2.43}$$

and noting that for $k_1 = k_2 = k_3 = k_4 = 0$ the terms on the right-hand side of equation (2.42) with $m = 1$ and $m = 2$ cancel out, we obtain

$$\Delta^{\Lambda}_{\mu_1\mu_2\mu_3\mu_4}(k_1, k_2, k_3, k_4) = \frac{-\mathrm{i}e^4}{(2\pi)^{12}4!}\sum_{\pi\{1,\ldots,4\}} \times \int_{\mathcal{D}(\Lambda)} \mathrm{d}^4 p\, \mathrm{tr}\big[S^{(3)}_{\mu_1\mu_2\mu_3}(p; 0, 0, 0)\gamma_{\mu_4}\big] + O(\Lambda^{-1}). \tag{2.44}$$

Thus, the fourth rank gauge-invariant polarization function has the familiar form [9]

$$K^{\Lambda}_{\mu_1\mu_2\mu_3\mu_4}(k_1, k_2, k_3, k_4) = \Pi^{\Lambda}_{\mu_1\mu_2\mu_3\mu_4}(k_1, k_2, k_3, k_4) - \Pi^{\Lambda}_{\mu_1\mu_2\mu_3\mu_4}(0, 0, 0, 0) + O(\Lambda^{-1}), \tag{2.45}$$

which converges when $\Lambda \to \infty$ and therefore is independent of the cutoff mode.

From the foregoing discussion, we conclude the following.

(1) The momentum-cutoff regularized fermion effective action $W_{G_\Lambda}[A]$ is a gauge-invariant functional of A_μ and obeys Furry's theorem.
(2) When $\Lambda \to \infty$, all the perturbation expansion coefficients of $W_{G_\Lambda}[A]$, except the logarithmically divergent term $\Pi^{\mathrm{div}}_{\mu\upsilon}(\Lambda, k)$ contained in the second-order coefficient $K^{\Lambda}_{\mu\upsilon}(-k, k)$, converge and agree with the conventional results obtained by using Pauli–Villars or dimensional regulators. On the other hand, we found that the asymptotic behaviour of $\Pi^{\mathrm{div}}_{\mu\upsilon}(\Lambda, k)$ is dependent on the cutoff mode $\mathcal{D}(\Lambda)$.
(3) In the modern QED theory, this divergent term should be cancelled out via a charge renormalization procedure. If we use the Lorentz-invariant cutoff $\mathcal{D}_4(\Lambda)$ in the calculation of $\Pi^{\mathrm{div}}_{\mu\upsilon}(\Lambda, k)$, there will be no problem in implementing the conventional renormalization procedure. However, if we want to use a 3D momentum cutoff such as $\mathcal{D}_3(\Lambda)$, an additional finite term of non-Lorentz-covariant character appears in $\Pi^{\mathrm{div}}_{\mu\upsilon}(\Lambda, k)$, which brings about new problems to the renormalization procedure.

3. Radiative corrections and Ward–Takahashi identity

In this section, we consider the radiative corrections due to the quantum fluctuations of virtual photons and electron–positron pairs in a momentum-cutoff vacuum. As we have shown in the introduction, for a momentum-cutoff vacuum, the vacuum persistence amplitude in the presence of an external electromagnetic field is assumed to be given by equation (1.13). Denoting the vacuum expectation of an arbitrary functional $F[A]$ by $F'[A]$, we have

$$F'[A] = \frac{1}{\langle 0_+|0_-\rangle^A_\Lambda}\int [\mathrm{d}a^\Lambda] F[A+a]\exp\{\mathrm{i}(S[a] + W_{G_\Lambda}[A+a])\}. \tag{3.1}$$

The functional integral on the right-hand side of equation (3.1) can be evaluated with the Faddeev–Popov procedure by specifying a gauge, which leads to [7]

$$F'[A] = \frac{\mathrm{e}^{\hat{D}_\Lambda}\{F[A]\,\mathrm{e}^{\mathrm{i}W_{G_\Lambda}[A]}\}}{\mathrm{e}^{\hat{D}_\Lambda}\{\mathrm{e}^{\mathrm{i}W_{G_\Lambda}[A]}\}}, \tag{3.2}$$

where $\hat{D}_\Lambda$ denotes a bilinear functional derivative operator

$$\hat{D}_\Lambda = \frac{(2\pi)^4}{2}\int_{\mathcal{D}(\Lambda)} \mathrm{d}^4k\, D^{\mu\nu}_{\mathrm{F}}(k)\frac{\delta}{\delta A^\mu(k)}\frac{\delta}{\delta A^\nu(-k)} \tag{3.3}$$

and $D^{\mu\nu}_{\mathrm{F}}(k)$ the photon propagator in the specified gauge. Thus, by taking account of the radiative corrections in a momentum-cutoff vacuum, the electron propagator in the presence of an external field should has the form

$$S'_{\rm F}[A] = \frac{{\rm e}^{\hat{D}_\Lambda}\{S_{\rm F}[A]\,{\rm e}^{{\rm i}W_{G_\Lambda}[A]}\}}{{\rm e}^{\hat{D}_\Lambda}\{{\rm e}^{{\rm i}W_{G_\Lambda}[A]}\}}. \tag{3.4}$$

To show the effects of the momentum cutoff on radiative corrections, here we calculate the electron self-energy by making use of equation (3.4). In the absence of external field, the one-loop approximation of the electron propagator has the form

$$S'_{\rm F}[A]|_{A=0} = (1 + \hat{D}_\Lambda + \cdots)S_{\rm F}[A]|_{A=0}. \tag{3.5}$$

Equation (3.5) can be written in the momentum representation, which reads

$$\begin{aligned}\langle p'|S'_{\rm F}[A]|p\rangle|_{A=0} &= (2\pi)^4\delta^{(4)}(p'-p)S'_{\rm F}(\Lambda, p)\\ &= (2\pi)^4\delta^{(4)}(p'-p)[S_{\rm F}(p) - {\rm i}S_{\rm F}(p)\Sigma(\Lambda, p)S_{\rm F}(p) + \cdots],\end{aligned} \tag{3.6}$$

where

$$\Sigma(\Lambda, p) = \frac{{\rm i}({\rm i}e)^2}{2(2\pi)^4}\int_{\mathcal{D}(\Lambda)} {\rm d}^4k\, D_{\rm F}^{\mu\nu}(k)[\gamma_\mu S_{\rm F}(p-k)\gamma_\nu + \gamma_\nu S_{\rm F}(p+k)\gamma_\mu]. \tag{3.7}$$

We use the photon propagator in the Feynman gauge and add a small photon mass μ to regularize the infrared divergence, then equation (3.7) has the form

$$\Sigma(\Lambda, p) = \frac{2{\rm i}e^2}{(2\pi)^4}\int_{\mathcal{D}(\Lambda)} {\rm d}^4k \frac{\not{p} + \not{k} - 2m_0}{\left[(p+k)^2 - m_0^2 + {\rm i}\epsilon\right](k^2 - \mu^2 + {\rm i}\epsilon)}. \tag{3.8}$$

What is of interest here is the dependence of the asymptotic behaviour of $\Sigma(\Lambda, p)$ on the momentum-cutoff mode. Note that the integration of k has been restricted on $\mathcal{D}(\Lambda)$ so that one should be especially cautious in calculating the linearly divergent part of the integral in equation (3.8). In fact, when we use the Lorentz-invariant cutoff $\mathcal{D}_4(\Lambda)$, the result obtained is the same as that using the Pauli–Villars cutoff,

$$\begin{aligned}\Sigma_{\mathcal{D}_4}(\Lambda, p) = \frac{e^2}{8\pi^2}\Bigg\{&\int_0^1 {\rm d}s[2m_0 - (1-s)\not{p}]\ln\frac{\Lambda^2}{sm_0^2 - s(1-s)p^2 + (1-s)\mu^2}\\ &+ \frac{1}{4}\not{p} - 2m_0\Bigg\} + O(\Lambda^{-1}),\end{aligned} \tag{3.9}$$

while when we use the cutoff $\mathcal{D}_3(\Lambda)$ in an inertial system specified by n^μ, we obtain

$$\begin{aligned}\Sigma_{\mathcal{D}_3}(\Lambda, p) = \frac{e^2}{8\pi^2}\Bigg\{&\int_0^1 {\rm d}s[2m_0 - (1-s)\not{p}]\ln\frac{\Lambda^2}{sm_0^2 - s(1-s)p^2 + (1-s)\mu^2}\\ &+ \left(\frac{2}{3} - \ln 2\right)\not{p} - 4(1-\ln 2)m_0 + \frac{1}{3}(pn)(\gamma n)\Bigg\} + O(\Lambda^{-1}).\end{aligned} \tag{3.10}$$

By comparing these two results, we see that besides the meaningless finite shifts of the mass and wavefunction renormalization constants, the 3D momentum cutoff $\mathcal{D}_3(\Lambda)$ also produces a non-Lorentz-covariant term within the self-energy part $\Sigma(\Lambda, p)$, which could be recognized as

$$\delta\Sigma(p) = \frac{e^2}{24\pi^2}(pn)(\gamma n). \tag{3.11}$$

Furthermore, we also expect that there will be a Lorentz anomalous term in one-loop vertex correction if we use the $\mathcal{D}_3(\Lambda)$ cutoff procedure. In fact, in an external field, the vertex operator is related to the electron propagator by the formula

$$\Gamma_\mu(k, A) = \frac{{\rm i}(2\pi)^4}{e}\frac{\delta}{\delta A^\mu(k)}(S'_{\rm F}[A])^{-1}. \tag{3.12}$$

Since

$$\langle p'|(S'_{\mathrm{F}}[A])^{-1})|p\rangle|_{A=0} = (2\pi)^4\delta^{(4)}(p'-p)S'^{-1}_{\mathrm{F}}(\Lambda, p), \tag{3.13}$$

we have

$$\langle p'|\Gamma_\mu(k, A)|p\rangle|_{A=0} = -\frac{\mathrm{i}(2\pi)^4}{e}S'^{-1}_{\mathrm{F}}(\Lambda, p')\ \langle p'|\frac{\delta S'_{\mathrm{F}}[A]}{\delta A^\mu(k)}|p\rangle\bigg|_{A=0} S'^{-1}_{\mathrm{F}}(\Lambda, p). \tag{3.14}$$

By making use of the one-loop approximation,

$$S'^{-1}_{\mathrm{F}}(\Lambda, p) = S^{-1}_{\mathrm{F}}(p) + \mathrm{i}\Sigma(\Lambda, p) + \cdots, \tag{3.15}$$

$$\begin{aligned}\langle p'|\frac{\delta S'_{\mathrm{F}}[A]}{\delta A^\mu(k)}|p\rangle\bigg|_{A=0} &= \mathrm{ie}\,\delta^{(4)}(p'-p-k)\bigg[S^{(1)}_\mu(p;k)\\ &\quad - \frac{3e^2}{(2\pi)^4}\int_{\mathcal{D}(\Lambda)} \mathrm{d}^4q\, D^{\nu\lambda}_{\mathrm{F}}(q)\bar{S}^{(3)}_{\mu\nu\lambda}(p;k,q,-q) + \cdots\bigg],\end{aligned} \tag{3.16}$$

we obtain

$$\langle p'|\Gamma_\mu(k, A)|p\rangle|_{A=0} = (2\pi)^4\delta^{(4)}(p'-p-k)[\gamma_\mu + \Lambda_\mu(\Lambda; p+k, p) + \cdots], \tag{3.17}$$

where

$$\Lambda_\mu(\Lambda; p', p) = -\frac{e^2}{(2\pi)^4}\int_{\mathcal{D}(\Lambda)} \mathrm{d}^4q\, D^{\nu\lambda}_{\mathrm{F}}(q)\gamma_\nu S_{\mathrm{F}}(p'+q)\gamma_\mu S_{\mathrm{F}}(p+q)\gamma_\lambda. \tag{3.18}$$

Now we separate out the ultraviolet divergent part of $\Lambda_\mu(\Lambda; p', p)$ and write

$$\Lambda_\mu(\Lambda; p', p) = \Lambda^{\mathrm{div}}_\mu(\Lambda) + \Lambda^c_\mu(p', p) + O(\Lambda^{-1}), \tag{3.19}$$

where

$$\begin{aligned}\Lambda^{\mathrm{div}}_\mu(\Lambda) &= \Lambda_\mu(\Lambda; p, p)|_{\not p = m_0}\\ &= \frac{4\mathrm{ie}^2}{(2\pi)^4}\int_0^1 s\,\mathrm{d}s\int_{\mathcal{D}(\Lambda)}\mathrm{d}^4q\,\frac{(\not p+\not q)\gamma_\mu(\not p+\not q) + m_0^2\gamma_\mu - 4m_0(p_\mu+q_\mu)}{\left[(q+sp)^2 + s(1-s)p^2 - sm_0^2 - (1-s)\mu^2\right]^3}\Bigg|_{\not p=m_0}.\end{aligned} \tag{3.20}$$

When we use the $\mathcal{D}_3(\Lambda)$ cutoff, we obtain

$$\begin{aligned}\Lambda^{\mathrm{div}}_\mu[\mathcal{D}_3(\Lambda)] &= \frac{e^2}{8\pi^2}\left\{\int_0^1 s\,\mathrm{d}s\left[\ln\frac{4\Lambda^2}{s^2m_0^2 + (1-s)\mu^2} + \frac{(s^2+2s-2)m_0^2}{s^2m_0^2+(1-s)\mu^2}\right]\gamma_\mu\right.\\ &\quad\left. - \frac{7}{6}\gamma_\mu - \frac{1}{3}(n\gamma)n_\mu\right\} + O(\Lambda^{-1}).\end{aligned} \tag{3.21}$$

With a little manipulation of the result given by equation (3.10), it is straightforward to verify the Ward identity

$$\Lambda^{\mathrm{div}}_\mu[\mathcal{D}_3(\Lambda)] = -\left.\frac{\partial\Sigma_{\mathcal{D}_3}(\Lambda, p)}{\partial p^\mu}\right|_{\not p=m_0}. \tag{3.22}$$

Finally, we turn to derive the Ward–Takahashi identity valid for an arbitrary momentum-cutoff vacuum. We note that, from equation (2.1) and the identity (2.16), it follows that

$$\mathrm{i}k^\mu\frac{\delta}{\delta A^\mu(k)}\langle p'|S_{\mathrm{F}}[A]|p\rangle = \frac{\mathrm{ie}}{(2\pi)^4}\{\langle p'|S_{\mathrm{F}}[A]|p+k\rangle - \langle p'-k|S_{\mathrm{F}}[A]|p\rangle\}. \tag{3.23}$$

After taking account of the radiative corrections due to quantum fluctuations in a momentum-cutoff vacuum, equation (3.23) has the form

$$\mathrm{i}k^{\mu}\frac{\delta}{\delta A^{\mu}(k)}\langle p'|S'_{\mathrm{F}}[A]|p\rangle = \frac{\mathrm{i}e}{(2\pi)^4}\{\langle p'|S'_{\mathrm{F}}[A]|p+k\rangle - \langle p'-k|S'_{\mathrm{F}}[A]|p\rangle\}. \tag{3.24}$$

By making use of equation (3.24), we derive from equation (3.14) the Ward–Takahashi identity

$$-\mathrm{i}k^{\mu}\langle p'|\Gamma_{\mu}(k,A)|p\rangle|_{A=0} = (2\pi)^4\delta^{(4)}(p'-p-k)\big[S'^{-1}_{\mathrm{F}}(\Lambda,p+k) - S'^{-1}_{\mathrm{F}}(\Lambda,p)\big]. \tag{3.25}$$

4. Discussion and summary

In the introduction, we have proposed to use equation (1.13) as a basis for constructing a momentum-cutoff model of QED which could preclude the ultraviolet divergence while still preserving the gauge invariance of QED. However, to realize this aim, one has to face the following two problems.

The first problem is about the formalism of fermion fields. As we have shown in this paper, an essential step which enables us to preserve gauge invariance is to implement a momentum cutoff on a gauge-invariant electron propagator rather than on a gauge-covariant electron propagator. Perhaps, it is more reasonable to first develop a gauge-invariant formulation of fermion fields, regularize it by a momentum cutoff and then implement the quantization. In an earlier work, the author has studied a gauge-invariant Hamiltonian formalism for spinor electrodynamics [10], where the basic state variables used for describing fermions are gauge-invariant bilinear forms of Dirac fields. We hope that, by implementing a quantization procedure in such a formalism, we would obtain an effective action of fermion just like what we have assumed in equation (1.12).

The next problem we have to face is the violation of Lorentz symmetry due to a 3D momentum ultraviolet cutoff. It is obvious that there will be Lorentz-invariance violation when a momentum cutoff such as $\mathcal{D}_3(\Lambda)$ has been imposed on all the fields. But the real trouble arises only if the Lorentz-invariance violation survives after the regulator is removed (i.e. after $\Lambda \to \infty$). In fact, the calculation of a logarithmically divergent Lorentz-covariant integral with the $\mathcal{D}_3(\Lambda)$ cutoff will generally yield, besides a Lorentz-covariant divergent term, a non-Lorentz-covariant finite term. We shall call these non-Lorentz-covariant terms resulted from the ultraviolet cutoff as Lorentz anomalies.

In equations (2.40), (3.11) and (3.21), we have shown the QED Lorentz anomalies in the one-loop approximation for photon self-energy, electron self-energy and vertex correction, respectively. Since these anomalous terms have no obvious physical meaning, it seems that they bring about new troubles into the generally accepted QED renormalization theory. However, we note that these three anomalous terms are supplementary terms associated with the renormalization constants Z_3, Z_2 and Z_1, respectively, and therefore can be cancelled by adding extra counter terms in the QED Lagrangian. Further works are needed to show whether we could achieve a complete cancellation of the Lorentz anomalies in a 3D momentum-cutoff model of QED via a renormalization procedure. If we could find such a renormalization procedure, the Lorentz symmetry will be restored in the renormalized theory.

In summary, we have developed in this paper a formalism for QED which preserves gauge invariance when photon and electron propagators are regularized with a sharp momentum-cutoff procedure. When the cutoff is implemented on the Euclidean 4-momentum space, Lorentz symmetry can be preserved so that the momentum-cutoff procedure could be used as a convenient gauge-invariant regulator. We have also discussed the viewpoint which treats the momentum cutoff as a real cutoff imposed on the fields rather than a temporary measure in the procedure of renormalization. It was shown that when a logarithmically divergent Lorentz-covariant integral is regularized with a 3D momentum cutoff, a finite non-Lorentz-covariant term may show up as a by-product of the ultraviolet divergence.

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Appendix

The identities (2.14) and (2.15) may be derived from

$$\begin{aligned}
&\int_0^1 \mathrm{d}s_1 \cdots \int_0^1 \mathrm{d}s_m \,\mathrm{tr}\left[S^{(n-m)}_{\mu_{m+1}\cdots\mu_n}\left(p + \sum_{j=1}^{m} s_j k_j; k_{m+1}, \ldots, k_n \right) \gamma_\mu \right] \\
&\qquad = (-1)^{n-m+1} \int_0^1 \mathrm{d}s_1 \cdots \int_0^1 \mathrm{d}s_m \\
&\qquad\quad \times \mathrm{tr}\left[S^{(n-m)}_{\mu_{m+1}\cdots\mu_n}\left(-p - \sum_{j=1}^{m} s_j k_j; -k_{m+1}, \ldots, -k_n \right) \gamma_\mu \right] \\
&\qquad = (-1)^{n-m+1} \int_0^1 \mathrm{d}s_1 \cdots \int_0^1 \mathrm{d}s_m \\
&\qquad\quad \times \mathrm{tr}\left[S^{(n-m)}_{\mu_n\cdots\mu_{m+1}}\left(-p - \sum_{j=1}^{n} k_j + \sum_{j=1}^{m} s_j k_j; k_n, \ldots, k_{m+1} \right) \gamma_\mu \right].
\end{aligned} \tag{A.1}$$

It is easy to verify the identity (2.16) for the case $n = 1$. We use induction to give a proof for the cases $n > 1$. From definitions (2.2) and (2.12), we may write

$$\begin{aligned}
\bar{S}^{(n+1)}_{\mu\mu_1\cdots\mu_n}(p; k, k_1, \ldots, k_n) &= \frac{1}{n+1}\Bigg[\bar{S}^{(n)}_{\mu_1\cdots\mu_n}(p + k; k_1, \ldots, k_n)\gamma_\mu S_{\mathrm{F}}(p) \\
&\quad + \sum_{j=1}^{n} \bar{S}^{(n)}_{\mu\mu_1\cdots\check{\mu}_j\cdots\mu_n}(p + k_j; k, k_1, \ldots, \check{k}_j, \ldots, k_n)\gamma_{\mu_j} S_{\mathrm{F}}(p)\Bigg],
\end{aligned} \tag{A.2}$$

where we use the symbol $\check{x}$ to denote the absence of x. We observe that

$$\begin{aligned}
&\mathrm{i}k^\mu \bar{S}^{(n)}_{\mu_1\cdots\mu_n}(p + k; k_1, \ldots, k_n)\gamma_\mu S_{\mathrm{F}}(p) \\
&\qquad = \bar{S}^{(n)}_{\mu_1\cdots\mu_n}(p + k; k_1, \ldots, k_n) - \bar{S}^{(n)}_{\mu_1\cdots\mu_n}(p; k_1 + k, k_2, \ldots, k_n).
\end{aligned} \tag{A.3}$$

On the other hand, (2.16) implies that

$$\begin{aligned}
&\mathrm{i}k^\mu \sum_{j=1}^{n} \bar{S}^{(n)}_{\mu\mu_1\cdots\check{\mu}_j\cdots\mu_n}(p + k_j; k, k_1, \ldots, \check{k}_j, \ldots, k_n)\gamma_{\mu_j} S_{\mathrm{F}}(p) \\
&\qquad = \frac{1}{n}\sum_{j=1}^{n}\Big[\bar{S}^{(n-1)}_{\mu_1\cdots\check{\mu}_j\cdots\mu_n}(p + k_j + k; k_1, \ldots, \check{k}_j, \ldots, k_n) \\
&\qquad\quad - \bar{S}^{(n-1)}_{\mu_1\cdots\check{\mu}_j\cdots\mu_n}(p + k_j; k_1, \ldots, \check{k}_j, \ldots, k_n)\Big]\gamma_{\mu_j} S_{\mathrm{F}}(p) \\
&\qquad = \bar{S}^{(n)}_{\mu_1\cdots\mu_n}(p; k_1 + k, k_2, \ldots, k_n) - \bar{S}^{(n)}_{\mu_1\cdots\mu_n}(p; k_1, k_2, \ldots, k_n).
\end{aligned} \tag{A.4}$$

Thus, we obtain

$$\mathrm{i}(n+1)k^{\mu}\bar{S}^{(n+1)}_{\mu\mu_1\cdots\mu_n}(p;k,k_1,\ldots,k_n)$$
$$=\bar{S}^{(n)}_{\mu_1\cdots\mu_n}(p+k;k_1,\ldots,k_n)-\bar{S}^{(n)}_{\mu_1\cdots\mu_n}(p;k_1,\ldots,k_n). \tag{A.5}$$

The identity (2.17) may be regarded as a consequence of the gauge invariance of the propagator $G[A]$. However, as a check of our formalism, it is worth to present a direct proof. Let

$$\mathrm{i}k_1^{\mu_1}\sum_{\pi\{1,\ldots,n\}}\frac{\mathrm{i}^m}{m!}\int_0^1\mathrm{d}s_1\cdots\int_0^1\mathrm{d}s_m\,\partial_{\mu_1}\cdots\partial_{\mu_m}S^{(n-m)}_{\mu_{m+1}\cdots\mu_n}\left(p+\sum_{j=1}^m s_jk_j;k_{m+1},\ldots,k_n\right)$$
$$=A_m+B_m, \tag{A.6}$$

where A_m represents the sum of the terms with the factor $k_1^{\mu_1}\partial_{\mu_1}$ and B_m all the other terms. Observing that

$$k_1^{\mu_1}\partial_{\mu_1}F\left(p+\sum_{j=1}^m s_jk_j\right)=\frac{\mathrm{d}}{\mathrm{d}s_1}F\left(p+\sum_{j=1}^m s_jk_j\right), \tag{A.7}$$

we have

$$A_m=\frac{-\mathrm{i}^{m-1}}{(m-1)!}\sum_{\pi\{2,\ldots,n\}}\int_0^1\mathrm{d}s_2\cdots\int_0^1\mathrm{d}s_m\,\partial_{\mu_2}\cdots\partial_{\mu_m}\left[S^{(n-m)}_{\mu_{m+1}\cdots\mu_n}\left(p+k_1+\sum_{j=2}^m s_jk_j;k_{m+1},\ldots,k_n\right)\right.$$
$$\left.-S^{(n-m)}_{\mu_{m+1}\cdots\mu_n}\left(p+\sum_{j=2}^m s_jk_j;k_{m+1},\ldots,k_n\right)\right]. \tag{A.8}$$

By making use of the identity (2.16), we have

$$B_m=\frac{\mathrm{i}^m}{m!}\sum_{\pi\{2,\ldots,n\}}\int_0^1\mathrm{d}s_2\cdots\int_0^1\mathrm{d}s_m\,\partial_{\mu_2}\cdots\partial_{\mu_{m+1}}\left[S^{(n-m-1)}_{\mu_{m+2}\cdots\mu_n}\left(p+k_1+\sum_{j=2}^{m+1} s_jk_j;k_{m+2},\ldots,k_n\right)\right.$$
$$\left.-S^{(n-m-1)}_{\mu_{m+2}\cdots\mu_n}\left(p+\sum_{j=2}^{m+1} s_jk_j;k_{m+2},\ldots,k_n\right)\right]. \tag{A.9}$$

It follows that

$$A_m=-B_{m-1}. \tag{A.10}$$

Since

$$A_0=B_n=0, \tag{A.11}$$

the identity (2.17) is proved.

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